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The Fuzzy Threshold Voting System for Cryptocurrency Governance

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Abstract We introduce a mathematical model for a voting system called fuzzy threshold voting (FTV) intended for use primarily in cryptocurrency governance systems. The main difference between this voting model and approval voting is that the voting outcome depends not only on the scores of candidates but also on other factors—budgets of the corresponding projects (actually, projects are the candidates). The proposed FTV voting model is considered to be some generalization of the yes-no voting model with expanded features. In addition to building an adequate mathematical model for the new voting system, we also analyze its main properties, such as the optimality (in some sense)

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of the voting procedure, existence of admissible strategies, advantages of sincere voting, and necessity of secret voting. We demonstrate that the presented FTV system satisfies one of the most important properties of the voting systems—the monotone rule, and prove several statements regarding possible voting strategies.

Keywords voting system; fuzzy threshold voting; approval voting; yes-no voting; cryptocurrency

1 Introduction

Since the mid-20th century, voting systems have been a subject of interest for economists and mathematicians worldwide. There are a lot of papers devoted to attempts to design the perfect voting system. Primarily, the scope of voting systems is political elections; however, voting theory has become increasingly necessary for certain information technology purposes. One such purpose is the design of a self-funded cryptocurrency governance (treasury) system, where participants vote on different projects (proposals) aiming at cryptocurrency improvement and growth. For example, such treasury systems are developed in Cardano (catalyst project [1]) and dash [2]. Both systems use a variation of approval voting. In such self-governing and self-funding ecosystems, control is distributed among community members rather than concentrated in a central authority. The long-term sustainability and innovative capacity of these networks depend critically on robust governance mechanisms capable of effectively allocating resources from a common treasury toward projects that foster their growth and development.

There is a wide variety of voting systems: Plurality [3], preferential systems [4–6], cardinal systems [7], etc. They have different approaches and properties but all try to meet the same goal—to satisfy all voters as much as possible. It is a difficult task because voters may act strategically to increase the likelihood of their satisfaction. The specific features of a cryptocurrency treasury system such as floating multiple numbers of winning proposals and the availability of a budget for each proposal (the amount of funds needed for its implementation), complicate the model and require a more rigorous approach when choosing a voting system. The challenge in a decentralized treasury system is not merely to pick the “most popular” project, but to solve a complex resource allocation problem: How to select an optimal portfolio of diverse projects that collectively maximize value for the ecosystem, all while adhering to strict and often fluctuating budgetary constraints. This requires a new class of voting models, specifically tailored to the unique environment of on-chain governance.

1.1 Related Work

There is a variety of different voting systems [8–18]. Voting models used in cryptocurrency governance are often based on a so-called approval voting (AV) system that was first described by Ottewell [8] and fully published by Brams and Fishburn in 1978 [9]. A classical model supposes that a voter may “approve” any number of candidates in contrast to plurality voting, where a voter should choose only one candidate, or preferential voting, where a voter should rank all candidates in some way. Voting models for governance systems usually have a slight modification compared to classical approval voting. It allows voting not only for “approval”, but also for "disapproval". So, in general, a voter has 3 possible choices for each proposal: "Yes", "No", and "Abstain". In a classical approval voting, he has only 2 choices: "Yes", or "Abstain", where "Abstain" is understood as a negative choice, while in the modified version it is a neutral choice. Moreover, in such systems there is a multiple-winner voting, so the proposals with the biggest amount of votes are elected and paid.

Such a system gives more possibilities for voters to express their preferences over candidates and tends to elect more optimal candidates compared to the classical AV schemes. The model is quite simple both in implementation and understanding, the results of an election are usually well-understood and accepted by the voters. It requires less effort from a voter to make a voting ballot because it does not necessarily require ranking candidates in some way (in contrast to preferential voting). A voter, for example, can only cast a vote for the favorite candidate and not consider other candidates.

On the other hand, a floating number of winners leads to many difficulties for the formal analysis from the game-theoretical point of view, as the system becomes tangled. The Condorcet criterion does not hold in the general case: There are some cases when a Condorcet winner may not be elected, and a condorcet loser may be elected as a winner. That leads to a non-optimal outcome of the election.

The monotone rule is also a critical criterion in voting systems that ensures fairness and consistency in electoral outcomes[19]. It states that if a candidate's position improves in the preferences of voters, their chances of winning should not decrease. By adhering to monotonicity, voting systems reduce the likelihood of strategic voting, where voters might misrepresent their preferences to influence the outcome[20].

Fuzzy decision-making is a process that allows you to make choices in situations involving uncertainty or imprecise information. It offers more flexible and nuanced decision-making compared to traditional methods. For example, the fuzzy version of the preferential voting model is proposed in [21]. This approach involves engaging with experts and decision-makers to assess the suitability of each option or criterion in various ranks. A fuzzy weighted voting along with blockchain technology is presented in [22]. According to the authors, such a combination provides a solution that ensures greater satisfaction among voters, helping people select the most effective and appropriate representative.

1.2 Our Contribution

In this paper, we introduce a mathematical model for a voting system that is used mainly in cryptocurrencies[23]. We call it fuzzy threshold voting (FTV). Note that our model has nothing to do with fuzzy logic. The word "fuzzy" emphasizes that a set of winners in some cases may contain candidates with lower scores and may not contain candidates with higher scores. This property is a result of the fact that voting outcome depends not only on the scores of candidates (i.e., on their rankings in the voters' ballots) but also on other factors—budgets of corresponding projects (actually projects are the candidates). This is the main difference between our voting model in comparison with the AV (as with other voting models). On one hand, this is also the main problem of the FTV model, as all existing theoretical analyses for voting models fail in this case. On the other hand, knowledge of projects' budgets gives voters some additional information that helps them to make more considerate decisions during the voting or to form more accurate preferences.

The proposed FTV voting model is considered to be some generalization of the yes-no voting model[24] with expanded features.

Two-factor dependence of the voting procedure and its fuzzy threshold makes the theoretical analysis of this model much more complicated. But despite this, we managed to obtain some results that may be considered analogies of the main results for the classical AV model[25,26].

In this paper, in addition to building of an adequate mathematical model for the new voting system, we also analyzed its main properties, such as the optimality (in some sense) of the voting procedure, existence of admissible strategies, advantages of sincere voting, and necessity of secret voting. We also show that, as for many voting systems, the Condorcet rule, generally speaking, does not hold for our model. However, we managed to find some additional conditions for a voting profile under which a pairwise winner always wins the voting.

2 Definition of FTV Model

In what follows, we will mainly use terms and definitions from[25]. We denote by I the finite set of voters and by X the finite set of projects (alternatives, candidates) and for some $n \in N$: $L^{(n)} = \{-n, -(n-1), \cdots, -1, 0, 1, \cdots, n-1, n\}$. The set $L^{(n)}$ contains all possible rankings that voters may give to projects. We assume $\#I \geq 2$ and $\#X \geq 2$. Every voter $i \in I$ has a preference over X expressed by a utility function $u_i : X \longrightarrow R$. In some particular cases, we will assume that $u_i : X \longrightarrow L^{(n)}$, i.e., the set of possible preferences coincides with the set of projects' rankings. For example, Proposition 3 that shows optimality of the voting procedure in some sense, is proved just with such restrictions. But these restrictions are quite appropriate and much closer to reality than, for example, in the satisfaction AV model[27] or the proportional AV model[28], because our model seriously takes into account the preference levels of all voters.

Given two projects $x, y \in X$, the voter i finds x at least as good as y if $u_i(x) \leq u_i(y)$. A project x is high in u_i if for all $y \in X$ $u_i(x) \geq u_i(y)$. We say that x is low in u_i if for all $y \in X$ $u_i(x) \leq u_i(y)$. We call u_i null whenever i is indifferent among all alternatives, i.e., $u_i(x) \geq u_i(y)$ for every $x, y \in X$. If u_i is null, then every project is both low and high in u_i. If u_i is not null then the projects that are higher in u_i and those that are lower in u_i form disjoint sets. We assume that each project the voter likes has some positive "rating" in his preferences, and each project the voter dislikes has some negative "rating".

A ballot B_i of the voter i is any partition of the set X on $2n+1$ subsets $H_i^{(l)}$, $l \in L^{(n)}$, which satisfy the following conditions:

1) $H_i^{(l)} \cap H_i^{(k)} = \varnothing$ for any $l, k \in L^{(n)}$, $l \neq k$;

2) $\cup_{l \in S} H_i^{(l)} = X$.

Note that one or more of subsets $H_i^{(l)}$, $l \in L^{(n)}$, may be empty.

It follows immediately from this definition that the number of possible ballots (or the number of voting strategies) for each voter is equal to $(\#X)^{2n+1}$.

When the voter i, $i \in I$, casts a ballot $B_i = (H_i^{(-n)}, \cdots, H_i^0, \cdots, H_i^{(n)})$, we say that i approves the projects in $H_i^{(l)}$, where $l > 0$, with the (positive) score l, abstains from the vote on the projects in H_i^0 and rejects the projects in $H_i^{(l)}$, where $l < 0$, with the (negative) score l.

For each voter i, $i \in I$, who casts the ballot $B_i = (H_i^{(-n)}, \cdots, H_i^0, \cdots, H_i^{(n)})$, we define his score function $s_i : X \longrightarrow L$, where $s_i(x) = l$ iff $x \in H_i^{(l)}$.

The FTV model is rather similar to the approval voting (AV) model but also has some essential differences. The first difference is in the definition of a ballot. In the AV model, a ballot B_i consists of only approved candidates, i.e. candidates from $H_i^{(l)}$ for a positive l. And all these candidates have the same "rating" on the ballot despite a voter's preferences. We also may say that in the AV model $n = 1$ and $H_i^{(0)} = \varnothing$ for each $i \in I$ and $H_i^{(-1)} = X \backslash H_i^{(1)}$. The other differences we will see below are in the definition of the score function and the definition of the voting procedure.

For the AV model, the following natural restrictions are usually made[25,26] to restrict the set of voting strategies that a voter is assumed to use:

1) It is assumed that voters use undominated strategies that are often called "admissible strategies" and can be characterized as follows: If a voter's preference is strict, he approves his preferred candidate, he does not approve his worst candidate and no constraint is imposed to other, intermediate candidates.

2) The other restriction is sincerity requirement imposing that when the voter approves a candidate, he also approves all the candidates she strictly prefers to this one.

But our model is rather different from the AV. We will not restrict strategies in the same way, but we will use some definitions of sincere voting for FTV. Let the voter i have some definite preferences on the set X of all projects.

Definition 1 Let $u_i : X \longrightarrow L^{(n)}$. We say that the voter i votes sincerely (in the narrow sense) if for any $x \in X$ the equality holds:

$$u_i(x) = \sum_{l \in L} l \cdot \delta(x, H_i^{(l)}),$$

where

$$\delta(a, A) = \begin{cases} 1, & \text{if } a \in A, \\ 0, & \text{otherwise.} \end{cases}$$

Definition 2 We say that the voter i votes sincerely (in the wide sense) for two projects $x, y \in X$, if the inequality

$$u_i(x) > u_i(y)$$

involves inequality $s_i(x) \geq s_i(y)$ (or involves $x \in H_i^{(l)}$, $y \in H_i^{(j)}$ for some $l \geq j$).

We say that the voter i votes sincerely (in a wide sense) if he votes sincerely for any $x, y \in X$.

Definition 3 We will call the set $B = (B_i)_{i\in I}$ of all ballots a ballot profile and for all $i \in I$ define $B_{-i} := B\backslash B_i = (B_j)_{j\in I\backslash\{i\}}$. We will write $B = (B_i, B_{-i})$ whenever we wish to highlight the dependency of B concerning i-th ballot. We refer to B_{-i} as a ballot profile without i.

Definition 4 Given a profile B, the score of project $x \in X$ is

$$s(x, B) = \sum_{l\in L} l \cdot \#\{i \in I : x \in H_i^{(l)}\}.$$

Note that in the AV model, the score of candidate $x \in X$ is defined as

$$s(x, B) = \#\{i \in I : x \in B_i\},$$

– the number of ballots that candidate x belongs to.

Definition 5 Given a profile B, we say that the project $x \in X$ is acceptable candidate, if

$$s(x, B) \geq 0.1 \cdot |I|.$$

In this model, only acceptable candidates may be winners.

Definition 6 We set some positive value M and will call this value a common budget given for the realization of projects from the set X.

For each project $x \in X$, we set some positive value $m(x)$ that is the budget of the project x. In what follows, we will assume that $\forall x \in X : m(x) \leq M$.

Definition 7 We say that some subset $S \subset X$ exhausts the common budget M, if the following conditions hold:

1) $\sum_{x\in S} m(x) \leq M$;
2) If $S \neq X$ then $\forall y \in X\backslash S$: $\sum_{x\in S} m(x) + m(y) > M$.

Given a ballot profile B, among all (acceptable) projects $X = \{x_i\}_{i\in I}$ we have to choose the set of winners (winning projects) $W(B) \subset X$. On one hand, the winners must be the projects with the highest scores; on the other hand, the set of winners must exhaust the common budget M. The latter restriction defines one more difference between AV and FTV models.

Definition 8 Let $\#X = k$ and, given a ballot profile B, the sequences

$$x_1, \cdots, x_k, x_i \in X, \tag{1}$$

is organized in a such way that

$$s(x_1, B) \geq \cdots \geq s(x_k, B). \tag{2}$$

In what follows, we will assume that, given a ballot profile B, conditions (1), (2) hold.

Definition 9 (Voting procedure) To define the set $W(B) \subset X$ of winners (winning projects), we will use the following (iterative) voting procedure:

$$x_1 \in W(B), \tag{3}$$

$$\text{for } i = \overline{2, k} : x_i \in W(B) \Longleftrightarrow \sum_{j=1}^{i-1} m(x_j)\delta(x_j, W(B)) + m(x_i) \leq M.$$

We should emphasize that the FTV voting procedure (3) is the main difference between the FTV model from the AV model, because in the FTV case, the set of winners depends not only on the "scores" of candidates, but also on their budgets, and the latter dependence is very essential in forming of the set of winners. This dependence involves a lot of difficulties in the theoretical analysis of the FTV model. The main difference is that there may be two projects $x, y \in X$ such that under some profile B $s(x, B) > s(y, B)$ and $y \in W(B)$, but $x \notin W(B)$. This feature makes it impossible to prove some statements that usually hold for "classical" voting models. For this reason, sometimes we will use a simplification of the voting procedure to prove the main statements.

Definition 10 (Simplified voting procedure) Let (1), (2) hold. According to a simplified voting procedure, given a ballot profile B,

$$W(B) = \{x_1, x_2, \cdots, x_t\},$$

where

$$t = \max\left\{l \geq 1 : \sum_{i=1}^{l} m(x_i) \leq M\right\}. \tag{4}$$

Under this simplification, the set of winners $W(B)$ is such that if $x_i \in W(B)$ for some $i \geq 2$, then $x_j \in W(B)$ for all $j = \overline{1, i-1}$. But we should note that even in this case the threshold is still fuzzy and so is the number of winners. Also note that voting procedures described in the Definitions 9 and 10 are identical if and only if the ballot profile B is such that for some t $(1 \leq t \leq n)$ $\{x_1, \cdots, x_t\}$ exhausts the common budget M.

A detailed block diagram describing the various steps involved in the proposed framework is presented in Figure 1.

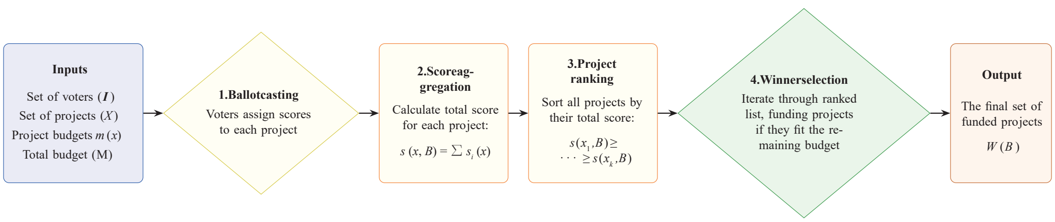

Figure 1 The steps of the voting procedure

Now let us prove some important properties of the voting procedure from the Definition 9.

Proposition 1 *(i) Under the voting procedure according to the Definition* 9, *if for some* $x \in X$ *the inequality*

$$\forall i \in I, \forall\ y \in X \backslash \{x\} : s_i(x) \geq s_i(y)$$

holds, then $x \in W(B)$.

(ii) The voting procedure from the Definition 9 *satisfies a monotone rule:*

a. if $x \in W(B)$ *and for some profile* $\tilde{B}$: $s(x,\tilde{B}) > s(x,B)$ *and* $s(y,\tilde{B}) = s(y,B)$ *for all* $y \in X\backslash\{x\}$, *then* $x \in W(\tilde{B})$;

b. if $x \notin W(B)$ *and for some profile* $\tilde{B}$: $s(x,\tilde{B}) < s(x,B)$ *and* $s(y,\tilde{B}) = s(y,B)$ *for all* $y \in X\backslash\{x\}$, *then* $x \notin W(\tilde{B})$.

Proof (i) This statement is obvious. Indeed, let $x \in X$ be such that

$$\forall i \in I \ \forall y \in X\backslash\{x\} : s_i(x) \geq s_i(y).$$

Then for any profile B and for any $y \in X\backslash\{x\}$,

$$s(x,B) = \sum_{i\in I} s_i(x) \geq \sum_{i\in I} s_i(y) = s(y,B),$$

so $s(x,B) = \max_{y\in X} s(y,B)$.

According to the Definition 6, $m(x) \leq M$. So under voting procedures (3) or (4) $x \in W(B)$ for any profile B. The first statement is proved.

(ii) Let for some profile B $x \in W(B)$ and ballot profile $\tilde{B}$ be such that

$$s(x,\tilde{B}) > s(x,B) \text{ and for any } y \in X\backslash\{x\}\text{: } s(y,B) = s(y,\tilde{B}).$$

Let projects $x_1, \cdots, x_n$ be written according to Definition 8. And let under ballot profile B $x = x_i$ for some $i = 1, \cdots, n$. As $x \in W(B)$, then $\sum_{j=1}^{i} m(x_j) \leq M$. If $s(x,\tilde{B}) > s(x,B)$, then under the ballot profile $\tilde{B}$ $x = x_{i-l}$ for some l such that $0 \leq l \leq i-1$. Then $\sum_{j=1}^{i-l} m(x_j) \leq \sum_{j=1}^{i} m(x_j) \leq M$, so $x \in W(\tilde{B})$.

Part a of the second statement is proven.

To prove part b, we can use the same considerations.

The proposition is proven.

Note 1 For $x, y \in X$, let us define

$$f(x,y;B) = \#\{B_i \in B : s_i(x) > s_i(y)\}.$$

Accordingly,

$$f(y,x;B) = \#\{B_i \in B : s_i(y) > s_i(x)\}.$$

Let for some $x \in X$ the following statement hold that is similar to the Condorcet winner definition:

$$\forall y \in X\backslash\{x\} f(x,y;B) > f(y,x;B). \tag{5}$$

The condition (5) means that for all $y \in X\backslash\{x\}$, x has a larger score than y for the majority of ballots.

Note that even under condition (5) we cannot guarantee that $x \in W(B)$. To illustrate this statement, let us consider the following example.

Let $n = 1$, and let us have two projects, $X = \{x, y\}$; $m(y), m(x) < M$.

Assume $I = 7$ and $B = (B_1, \cdots, B_7)$ where:

$$B_1 = B_2 = (x, y, \varnothing),$$
$$B_3 = B_4 = (\varnothing, x, y),$$
$$B_5 = (y, x, \varnothing),$$
$$B_6 = B_7 = (y, \varnothing, x).$$

Then

$$f(x, y; B) = \#\{B_1, B_2, B_3, B_4\} = 4,$$
$$f(y, x; B) = \#\{B_5, B_6, B_7\} = 3,$$

and the condition (5) holds.

But

$$s(x, B) = 1 + 1 + 0 + 0 + 0 - 1 - 1 = 0,$$
$$s(y, B) = 0 + 0 - 1 - 1 + 1 + 1 + 1 = 1,$$
$$s(y, B) > 0.1 \times 7 = 0.7,$$

so $y \in W(B)$, but $s(x, B) < 0.7$ and $x \notin W(B)$.

But we can formulate a stronger condition than condition (5) that would guarantee the "pairwise winner" to be the winner.

Proposition 2 *Let $n = 1$ and, for some $x \in X$, let the condition* (5) *hold. Then, if for some $y \in X$ the additional condition is satisfied,*

$$\#\{B_i \in B : s_i(x) = 1, s_i(y) = -1\} \geq \#\{B_i \in B : s_i(x) = -1, s_i(y) = 1\}, \tag{6}$$

then $s(x, B) > s(y, B)$.

Proof Let us define $a(u, v) = \#\{B_i \in B : s_i(x) = u, s_i(y) = v\}$, where $u, v \in \{-1, 0, 1\}$, and we rewrite the condition (6) as $a(1, -1) \geq a(-1, 1)$. Then

$$s(x) = a(1, 0) + a(1, -1) + a(1, 1) - a(-1, 1) - a(-1, 0) - a(-1, -1),$$
$$s(y) = a(0, 1) + a(-1, 1) + a(1, 1) - a(0, -1) - a(-1, -1) - a(1, -1)$$

and

$$\begin{aligned} s(x) - s(y) &= a(1, 0) + a(1, -1) - a(-1, 1) - a(-1, 0)a(0, -1) + a(1, -1) - a(0, 1) - a(-1, 1) \\ &= \{[a(1, 0) + a(1, -1) + a(0, -1)][a(-1, 0) + a(0, 1) + a(-1, 1)]\} + \\ &\quad \{a(1, -1) - a(-1, 1)\}. \end{aligned}$$

The first term in {} brackets is positive because of the condition (5), and the second one is non-negative because of the condition (6).

Then $s(x) - s(y) > 0$ or $s(x) > s(y)$.

Corollary 1 *Let $n=1$, and, for some $x \in X$, let the condition* (5) *hold. If for $\forall y \in X \backslash \{x\}$ the condition* (6) *holds, then*

$$\forall y \in X \backslash \{x\} : s(x, B) > s(y, B),$$

i.e., $s(x, B)$ is larger.

Note 2 In the case of $\forall x, y \in X : s(x; B) \neq s(y; B)$ the conditions (3) define the set W uniquely. However, tie voting may happen in the opposite case (like in our model). In what follows, we assume that in the case of tie voting some fair procedure is used.

We suppose that voters vote secretly by casting ballots, while the FTV model is used as the outcome function. So, we consider a normal form game, where a strategy set for any voter i is the set of all possible ballots (partitions of X). Hence a ballot profile B is also a strategy profile, and the outcome is the set of winning projects $W(B) \subset X$.

As $W(B)$ usually contains more than one project, our strategic analysis requires knowledge of voter's preferences over non-empty sets of X. We assume that ties over outcomes are broken according to Note 1 and that voters evaluate outcomes by expected Von-Neumann Morgenstern utilities. So, the utility that the voter i attaches to a set S of winning projects is

$$u_i(S) = \frac{1}{\#S} \sum_{x \in S} u_i(x). \tag{7}$$

Ideally, we wanted to choose the set $W(B)$ of winning projects in such a way that this set would maximize the average outcome

$$U(S) = \frac{1}{\#I} \sum_{i \in I} u_i(S). \tag{8}$$

But, generally speaking, it is not so for our voting procedure.

Nevertheless, we can set some conditions that are sufficient for W to maximize the function (8).

Proposition 3 *Let, given a ballot profile B and under conditions* (1), (2), (4), *$W = W(B) = \{x_1, \cdots, x_t\}$. Then, under the FTV sincere voting assumption (in the narrow sense):*

$$\max_{S \subset X : \#S \geq t} U(S) = U(W),$$

i.e. the average (on all voters) value of the satisfactory function, or the average outcome, takes its maximal value on the set of winners defined by the voting procedure (3)*.*

Proof Rewrite the function (8) using (7) in such way:

$$U(S) = \frac{1}{\#I} \sum_{i \in I} u_i(S) = \frac{1}{\#I} \sum_{i \in I} \left(\frac{1}{\#S} \sum_{x \in S} u_i(x) \right) = \frac{1}{\#I} \frac{1}{\#S} \sum_{x \in S} \left(\sum_{i \in I} u_i(x) \right). \tag{9}$$

It is easy to see that under the FTV sincere assumption and according to the definition of the score of the project

$$\sum_{i \in I} u_i(x) = s(x, B).$$

So we can rewrite the expression (9) as

$$U(S) = \frac{1}{\#I}\frac{1}{\#S}\sum_{x\in S}\left(\sum_{i\in I} u_i(x)\right) = \frac{1}{\#I}\frac{1}{\#S}\sum_{x\in S} s(x;B).$$

Now it is sufficient to prove that for any $S \subset X$, $\#S = f \geq t$, $S = \{s_1, \cdots, s_f\}$:

$$U(S) \leq U(W).$$

According to the definition of the function $U(S)$, is the same as

$$\frac{\sum_{i=1}^{f} s(s_i, B)}{f} \leq \frac{\sum_{i=1}^{t} s(x_i, B)}{t}, \tag{10}$$

considering that W is defined according to (4).

Define $T = S \cap W$, $\tilde{W} = W \backslash S$, $\tilde{S} = S \backslash W$, $f = t + v$ and

$$\Sigma_1 = \sum_{x\in T} s(x, B), \quad \Sigma_2 = \sum_{x\in \tilde{W}} s(x, B), \quad \Sigma_3 = \sum_{x\in \tilde{S}} s(x, B).$$

Let $|\tilde{W}| = z$, then $|\tilde{S}| = v + z$. Now we can rewrite (10) in an equivalent form:

$$\frac{\Sigma_1 + \Sigma_3}{t + v} \leq \frac{\Sigma_1 + \Sigma_2}{t},$$

or

$$t\Sigma_1 + t\Sigma_3 \leq t\Sigma_1 + v\Sigma_1 + t\Sigma_2 + v\Sigma_2,$$

or

$$t\Sigma_3 \leq v\Sigma_1 + t\Sigma_2 + v\Sigma_2. \tag{11}$$

According to the conditions of this proposition, $W = \{x_1, \cdots, x_t\}$, so $\tilde{S} \subset \{x_{t+1}, \cdots, x_k\}$, where $k = \#X$ and for any $x \in W$ and any $y \in \tilde{S}$: $s(x;B) < s(y;B)$. Also note that Σ_1 consists of $t - z$ items, Σ_2 consists of z items, and Σ_3 consists of $v + z$ items.

Then we can estimate the lefthand side of (11) as

$$t\Sigma_3 \leq t(v+z)\cdot \max_{x\in\tilde{S}} s(x,B) \leq t(v+z)\cdot \min_{x\in W} s(x,B).$$

Similarly, we can estimate the right part of (11) as

$$v\Sigma_1 + t\Sigma_2 + v\Sigma_2 \geq (tz + v(t-z) + vz)\cdot \min_{x\in W} s(x;B) = t(z+v)\cdot \min_{x\in W} s(x;B),$$

and (11) holds, then the equivalent inequality (10) also holds.

Corollary 2 *Let, given a ballot profile B and under the conditions* (1), (2), $W = W(B) = \{x_1, \cdots, x_t\}$.

Let also for any $S \subset X$ the following condition hold: If S exhausts the common budget M, then $\#S \geq t$.

Then, under the FTV sincere voting assumption (in the narrow sense):

$$\max_{S \subset X, S \text{ exhausts } M} U(S) = U(W).$$

This corollary means that under its conditions the set of winners chosen according to the model described by formula (3) *is the best choice to "satisfy" on average all voters.*

Note 3 Proposition 3 also holds when preferences of voters are proportional to the elements of L, i.e. when $\exists C > 0$, $\forall i \in I$, $\forall x \in X$: $u_i(x) \in C \cdot L = \{-Cn, \cdots, -C, 0, C, \cdots, Cn\}$.

3 Some Properties of FTV Model

Now we will discuss some properties of the AV model that we would like to be true (at least with some modification) for the FTV model. Firstly, we introduce some basic properties of the AV model and show, using a counterexample, that these properties do not hold for the FTV model.

Then we will state some other properties that may be considered, in some sense, as analogs of the AV model properties, and will prove that these properties hold for the FTV model.

First, we need some more definitions from Ref. [25].

Definition 11 For any voter i with preference u_i, we say that the ballot B_i (weakly) dominates the ballot $\tilde{B}_i$ if and only if

$$u_i(W(B_i, B_{-i})) \geq u_i(W(\tilde{B}_i, B_{-i}))$$

for all B_{-i}, and

$$u_i(W(B_i, B_{-i})) > u_i(W(\tilde{B}_i, B_{-i}))$$

for some B_{-i}.

A ballot is undominated if and only if it is dominated by no ballot.

Following[26], we also call undominated ballots admissible and use any of these words.

The following proposition characterizes admissible ballots for the AV model.

Proposition 4 (i) *If u_i is null, then all ballots are admissible for the voter i.*

(ii) *Let the number of voters be at least three. If u_i is not null then the ballot B_i is admissible for the voter i if and only if B_i contains every candidate who is high in u_i and no candidate who is low in u_i.*

The part (i) is trivial and follows directly from the definition.

The part (ii) is of great interest. Roughly speaking, it claims that it is better for a voter to vote "sincerely", or "according to her preferences", at least for the most preferable and the least preferable projects.

Now we show that part (ii), generally speaking, does not hold for the FTV model.

Example 1 Counterexample.

Let the set of voters $I = \{1, \cdots, 10\}$, the set of projects $X = \{x_1, \cdots, x_7, y\}$, $L = \{-2, -1, 0, 1, 2\}$, and $\forall i \in I$: $u_i(x) \in L$, as shown in Table 1.

Table 1 Preferences of the first voter

Project, x_i	x_1	x_2	x_3	x_4	x_5	x_6	x_7	y
$u_1(x_i)$	0	0	0	0	1	1	1	2

Define the common budget as $M = 7$ and the projects' budgets as

$$m(x_i) = 1, \quad i = \overline{1, 7}; \quad m(y) = 3.$$

Below we consider two options of voting for the first voter. In the first option he votes "sincerely" for the most preferable project y with the ballot $\tilde{B}_1$, i.e., the project y has the higher rating in $\tilde{B}_1$ (though it does not matter, without loss of generality we may assume that he also votes sincerely for other projects). In the second option, he votes oppositely with the ballot B_1, where all projects except y have the same rating as in ballot $\tilde{B}_1$, and the project y has the smallest rating. And we will show that for some B_{-1}:

$$u_1(W(B_1, B_{-1})) > u_1(W(\tilde{B}_1, B_{-1})).$$

Option 1: The voter votes for y according to his preferences.

For the ballot $\tilde{B}_1$, $y \in H_1^{(2)}$.

Let the other votes form the profile B_{-1} such that in profile $\tilde{B} = (\tilde{B}_1, B_{-1})$ the scores of projects are distributed according to Table 2.

Table 2 Scores of projects under the profile $\tilde{B} = (\tilde{B}_1, B_{-1})$

Project, x_i	x_1	x_2	x_3	x_4	y	x_5	x_6	x_7
$s(x_i, \tilde{B})$	8	7	6	5	4	3	2	1

In this case $W(\tilde{B}) = \{x_1, \cdots, x_4, y\}$ and $u_1(W(\tilde{B})) = \frac{2}{5} = \frac{14}{35}$.

Option 2: The voter votes for y against his preferences.

For ballot B_1, $y \in H_1^{(-2)}$.

Let other voters vote as in option 1, and let their votes form the same profile B_{-1}. Then in the profile $B = (B_1, B_{-1})$ the scores of projects are distributed according to the Table 3.

Table 3 Scores of projects under the profile $B = (B_1, B_{-1})$

Project, x_i	x_1	x_2	x_3	x_4	x_5	x_6	x_7	y
$s(x_i, B)$	8	7	6	5	3	2	1	0

In this case $W(B) = \{x_1, \cdots, x_7\}$ and $u_1(W(B)) = \frac{3}{7} = \frac{15}{35}$, i.e., $u_1(W(B)) > u_1(W(\tilde{B}))$.

After this example, one may conclude that roughly speaking, sincerity is not the best strategy in this model. But below we will see that, nevertheless, there exists some sense to vote sincerely.

Now we prove some statements for the FTV model that are particularly similar to the properties of the AV model proved in Proposition (4).

Proposition 5 *Under the voting procedure* (4), *the following statements hold:*

(i) *If u_i is null, then all ballots are admissible for the voter i.*

(ii) *If for some voter $i \in I$ and some project $y \in X$ the utility function u_i satisfies the inequality*

$$u_i(y) \geq \sum_{x \in X \setminus \{y\}} |u_i(x)|,$$

then the ballot B_i is admissible for voter i only if $y \in H_i^{(n)}$.

(iii) *If for some voter $i \in I$ and some project $y \in X$ the utility function u_i satisfies the inequality*

$$u_i(y) \leq -\sum_{x \in X \setminus \{y\}} |u_i(x)|,$$

then the ballot B_i is admissible for the voter i only if $y \in H_i^{(-n)}$.

Proof (i) The part (i) follows directly from the definition.

(ii) Consider a ballot $B_i = \{H_i^{(-n)}, \cdots, H_i^{(n)}\}$ for which $y \notin H_i^{(n)}$. It means that for the ballot B_i, $y \in H_i^{(v)}$ for some $-n \leq v \leq n-1$. Let $\tilde{B}_i$ be the ballot for which $\tilde{H}_i^{(v)} = H_i^{(v)} \setminus \{y\}$, $\tilde{H}_i^{(n)} = H_i^{(n)} \cup \{y\}$ and $\tilde{H}_i^{(j)} = H_i^{(j)}$ for all $j \in L \setminus \{v, n\}$. We will prove that $\tilde{B}_i$ dominates B_i.

Given any B_{-i}, all candidates except y have the same scores at (B_i, B_{-i}) and $(\tilde{B}_i, B_{-i})$, while the score of y is raised by $n - v$ units at the latter ballot profile. Therefore, regarding the sets of winning projects $Y = W(B_i, B_{-i})$ and $\tilde{Y} = W(\tilde{B}_i, B_{-i})$, the following three cases are exhaustive:

1. $y \notin Y$, $y \notin \tilde{Y}$;
2. $y \in Y$, $y \in \tilde{Y}$;
3. $y \notin Y$, $y \in \tilde{Y}$.

In the first two cases $Y = \tilde{Y}$ so $u_i(Y) = u_i(\tilde{Y})$.

In the third case, let $Y = \{x_1, \cdots, x_t\}$, then $\tilde{Y} = \{x_1, \cdots, y, \cdots, x_{t-l}\}$ for some $0 \leq l \leq t-1$. We show that in this case $u_i(\tilde{Y}) > u_i(Y)$. It is sufficient to show that for any l, $0 \leq l \leq t-1$,

$$\frac{\sum_{j=1}^{t} u_i(x_j)}{t} \leq \frac{\sum_{j=1}^{t-l} u_i(x_j) + u_i(y)}{t-l+1}. \tag{12}$$

Define

$$\Sigma_1 = \sum_{j=1}^{t-l} u_i(x_j), \quad \Sigma_2 = \sum_{j=t-l+1}^{t} u_i(x_j), \quad \Sigma = \sum_{j=1}^{t} u_i(x_j) = \Sigma_1 + \Sigma_2.$$

The inequality (12) is equivalent to the inequality

$$u_i(y) \geq \frac{\Sigma \cdot (t-l+1)}{t} - \Sigma_1 = \frac{\Sigma \cdot (t-l+1) - \Sigma_1 \cdot t}{t}$$

$$= \frac{\Sigma_2 \cdot (t-l+1) - \Sigma_1 \cdot (l-1)}{t}. \tag{13}$$

We estimate the right-hand part of inequality (13):

$$\begin{aligned}
\frac{\Sigma_2 \cdot (t-l+1) - \Sigma_1 \cdot (l-1)}{t} &\leq \frac{|\Sigma_2| \cdot (t-l+1) + |\Sigma_1| \cdot (l-1)}{t} \\
&\leq \frac{(t-l+1)\sum_{x \in X\setminus\{y\}} |u_i(x)| + (l-1)\sum_{x \in X\setminus\{y\}} |u_i(x)|}{t} \\
&= \sum_{x \in X\setminus\{y\}} |u_i(x)| \leq u_i(y),
\end{aligned}$$

where the latter inequality holds according to the conditions of the theorem. So inequality (13) holds and so does (12).

Now we have to show that there exists some profile B_{-i} that

$$u_i(W(\tilde{B}_i, B_{-i})) > u_i(W(B_i, B_{-i})).$$

Let profile B_{-i} be such that exactly $\lceil 0.1 \cdot \#I \rceil - v - 1$ voters vote with $y \in H_1$ and $z \in H_0$ for any $z \in X\setminus\{y\}$. All the remaining voters vote $z \in H_0$ for any $z \in X$. Then for $B = (B_i, B_{-i})$:

$$\begin{aligned}
&s(y, B) = \lceil 0.1 \cdot \#I \rceil - v - 1 + v = \lceil 0.1 \cdot \#I \rceil - 1, \\
&s(z, B) = 0, \quad \text{for any } z \in X\setminus\{y\},
\end{aligned}$$

so $W(B) = \varnothing$ and $u_i(W(B)) = 0$.

At the same time, for profile $\tilde{B} = (\tilde{B}_i, B_{-i})$:

$$\begin{aligned}
&s(y, \tilde{B}) = \lceil 0.1 \cdot \#I \rceil - v - 1 + n \geq \lceil 0.1 \cdot \#I \rceil \text{because } n \geq v - 1, \\
&s(z, B) = 0, \text{for any } z \in X\setminus\{y\},
\end{aligned}$$

so $W(\tilde{B}) = \{y\}$ and $u_i(W(\tilde{B})) = u_i(y)$.

Then, under the profile B_{-i}, $u_i(W(\tilde{B}_i, B_{-i})) > u_i(W(B_i, B_{-i}))$. Part (ii) is proven.

(iii) Consider a ballot $B_i = \{H_i^{(-n)}, \cdots, H_i^{(n)}\}$ for which $y \notin H_i^{(-n)}$. It means that for the ballot B_i, $y \in H_i^{(v)}$ for some $-(n-1) \leq v \leq n$. Let $\tilde{B}_i$ be the ballot for which $\tilde{H}_i^{(v)} = H_i^{(v)}\setminus\{y\}$, $\tilde{H}_i^{(-n)} = H_i^{(-n)} \cup \{y\}$ and $\tilde{H}_i^{(j)} = H_i^{(j)}$ for all $j \in L\setminus\{v, -n\}$. We will prove that $\tilde{B}_i$ dominates B_i.

Given any B_{-i}, all candidates except y have the same score at (B_i, B_{-i}) and $(\tilde{B}_i, B_{-i})$, while the score of y is decreased by $n + v$ units at the latter ballot profile. Therefore, regarding the sets of winning projects $Y = W(B_i, B_{-i})$ and $\tilde{Y} = W(\tilde{B}_i, B_{-i})$, the following three cases are exhaustive:

1. $y \notin Y$, $y \notin \tilde{Y}$;
2. $y \in Y$, $y \in \tilde{Y}$;
3. $y \in Y$, $y \notin \tilde{Y}$.

In the first two cases $Y = \tilde{Y}$, so $u_i(Y) = u_i(\tilde{Y})$.

In the third case, let $\tilde{Y} = \{x_1, \cdots, x_t\}$, then $Y = \{x_1, \cdots, y, \cdots, x_{t-l}\}$ for some $0 < l \leq t-1$. Show that in this case $u_i(\tilde{Y}) \geq u_i(Y)$. It is sufficient to show that for any l, $0 < l \leq t-1$,

$$\frac{\sum_{j=1}^{t} u_i(x_j)}{t} \geq \frac{\sum_{j=1}^{t-l} u_i(x_j) + u_i(y)}{t-l+1}. \tag{14}$$

Define

$$\Sigma_1 = \sum_{j=1}^{t-l} u_i(x_j), \quad \Sigma_2 = \sum_{j=t-l+1}^{t} u_i(x_j), \quad \Sigma = \sum_{j=1}^{t} u_i(x_j) = \Sigma_1 + \Sigma_2.$$

The inequality (14) is equivalent to inequality

$$\begin{aligned} u_i(y) &\leq \frac{\Sigma \cdot (t-l+1)}{t} - \Sigma_1 = \frac{\Sigma \cdot (t-l+1) - \Sigma_1 \cdot t}{t} \\ &= \frac{\Sigma_2 \cdot (t-l+1) - \Sigma_1 \cdot (l-1)}{t}. \end{aligned} \tag{15}$$

Estimate the right part of inequality (15):

$$\begin{aligned} \frac{\Sigma_2 \cdot (t-l+1) - \Sigma_1 \cdot (l-1)}{t} &\geq \frac{-|\Sigma_2| \cdot (t-l+1) - |\Sigma_1| \cdot (l-1)}{t} \\ &\geq \frac{-(t-l+1)\sum_{x\in X\backslash\{y\}} |u_i(x)| - (l-1)\sum_{x\in X\backslash\{y\}} |u_i(x)|}{t} \\ &= -\sum_{x\in X\backslash\{y\}} |u_i(x)| \geq u_i(y), \end{aligned}$$

where the last inequality holds according to the conditions of the theorem.

So inequalities (14), (15) hold.

Now we have to show that there exists some profile B_{-i} that

$$u_i(W(\tilde{B}_i, B_{-i})) > u_i(W(B_i, B_{-i})).$$

Let profile B_{-i} be such that exactly $\lceil 0.1 \cdot \#I \rceil + n - 1$ voters vote with $y \in H_1$, and $z \in H_0$ for any $z \in X\backslash\{y\}$. All the remaining voters vote $z \in H_0$ for any $z \in X$.

It is easy to see that in this case $W(\tilde{B}) = \varnothing$ and $u_i(W(\tilde{B})) = 0$.

At the same time, $W(B) = \{y\}$ and $u_i(W(B)) = u_i(y)$, so

$$u_i(W(\tilde{B}_i, B_{-i})) > u_i(W(B_i, B_{-i}))$$

and part (iii) is proved.

Proposition 6 *Let u_i be not null. Let for some $x, y \in X$ with $u_i(x) > u_i(y)$ the voter i votes sincerely with the ballot B_i such that $s_i(x) > s_i(y)$. Then:*

1) *if $u_i(x) > 0$, then there exists a profile B_{-i} such that for any ballot $\tilde{B}_i$ with $s'_i(x) < {s'}_i(y)$:*

$$u_i(W(B_i, B_{-i})) > u_i(W(\tilde{B}_i, B_{-i})), \tag{16}$$

2) *if* $u_i(x) \leq 0$*, then there exists a profile* B_{-i} *such that for any ballot* $\tilde{B}_i$ *with* $s'_i(x) < {s'}_i(y)$*:*

$$u_i(W(B_i, B_{-i})) \geq u_i(W(\tilde{B}_i, B_{-i})), \tag{17}$$

and for some $\widehat{B_i}$ *with* $s'_i(x) < {s'}_i(y)$*:*

$$u_i(W(B_i, B_{-i})) > u_i(W(\widehat{B_i}, B_{-i})). \tag{18}$$

Proof Let in the ballot B_i: $s_i(x) = a$, $s_i(y) = b$ $(a > b)$.

1) Consider the case when $u_i(x) > 0$.

Let profile B_{-i} be such that exactly $\lceil 0.1 \cdot \#I \rceil - a$ voters vote with $x, y \in H_1$ and $z \in H_0$ for any $z \in X \backslash \{x, y\}$. All the remaining voters vote $z \in H_0$ for any $z \in X$.

Then for $B = (B_i, B_{-i})$:

$$s(x, B) = \lceil 0.1 \cdot \#I \rceil - a + a = \lceil 0.1 \cdot \#I \rceil,$$
$$s(y, B) = \lceil 0.1 \cdot \#I \rceil - a + b < \lceil 0.1 \cdot \#I \rceil,$$

so

$$W(B) = \{x\},$$

and

$$u_i(W(B)) = u_i(x).$$

Let now $\tilde{B}_i$ be any other ballot with $s'_i(x) < s'_i(y)$. Let $s'_i(x) = c$, $s'_i(y) = d$. Define $\tilde{B} = (\tilde{B}_i, B_{-i})$. Then:

$$W(\tilde{B}) = \{\{x, y\}, \text{if } c \geq a; \{y\}, \text{if } c < a \text{ and } d \geq a; \{\varnothing\}, \text{if } d < a\}. \tag{19}$$

In all cases from (19), $u_i(W(\tilde{B})) < u_i(W(B))$, because of conditions $u_i(x) > 0$ and $u_i(x) > u_i(y)$ (we set $u_i(\varnothing) = 0$), and inequality (16) is proven.

2) Consider now the case when $u_i(x) \leq 0$.

Let profile B_{-i} be such that exactly $\lceil 0.1 \cdot \#I \rceil - a - 1$ voters vote with $x, y \in H_1$ and $z \in H_0$ for any $z \in X \backslash \{x, y\}$. All the remaining voters vote $z \in H_0$ for any $z \in X$.

Then for $B = (B_i, B_{-i})$:

$$s(x, B) = \lceil 0.1 \cdot \#I \rceil - a - 1 + a < \lceil 0.1 \cdot \#I \rceil,$$
$$s(y, B) = \lceil 0.1 \cdot \#I \rceil - a - 1 + b < \lceil 0.1 \cdot \#I \rceil,$$

so

$$W(B) = \varnothing.$$

and

$$u_i(W(B)) = 0.$$

Let now $\tilde{B}_i$ be any other ballot with $s_i'(x) < s'_i(y)$ and $s_i'(x) = c$, $s_i'(y) = d$. Then for profile $\tilde{B} = (\tilde{B}_i, B_{-i})$ the following options exist:

$$W(\tilde{B}) = \{\{x, y\}, \text{if } c > a; \{y\}, \text{if } c \leq a \text{and} d > a; \{\varnothing\}, \text{if } d \leq a\}. \tag{20}$$

In the first two cases from (20), $u_i(W(\tilde{B})) < u_i(W(B))$, and in the third case $u_i(W(\tilde{B})) = u_i(W(B))$. The inequalities (17), (18) are proven.

4 The Necessity of Secret Ballots

Now let us see how significant is to keep the results of the voting secret till the end of the voting procedure. Below we describe the situation when the voter i, who votes the last, can fully change the set of winners, especially knowing the profile B_{-i} with all other votes. One can construct a lot of more complicated examples that involve the same result.

In what follows, for some $x \in X$ we will denote as $s(x, B_{-i})$ the score of the project x under the profile B_{-i}, and as $W(B_{-i})$ the set of winners under the profile B_{-i}, i.e. the set of projects that won according to the voting procedure (3), when their scores are $s(x, B_{-i})$.

Proposition 7 *There exist such profile B_{-i}, such ballot B_i and such budgets distribution $m = \{m(x_i), i \in I\}$, that*

$$W(B_{-i}) \cap W(B_i, B_{-i}) = \varnothing,$$

(i.e., a voter $\boldsymbol{i}$, voting with B_i, can completely change the set of winners).

Proof To prove the statement, it is sufficient to give a relevant example. Let $n = 1$ and under profile B_{-i} let the sequences of projects (in descending order by score) be

$$x_1, \cdots, x_l,$$

where $s(x_1, B_{-i}) = s(x_2, B_{-i}) + 1$ and the budgets' distribution be:

$$m(x_1) = 0,5 \cdot M, \quad m(x_2) = M, \quad m(x_3) + \cdots + m(x_k) = 0,5 \cdot M \quad \text{for some } k \leq l.$$

Then, under the assumption that all candidates $x_1, \cdots, x_k$ are acceptable, and under the profile B_{-i} the set of winners is

$$W_1 = W(B_{-i}) = \{x_1, x_3, \cdots, x_k\}.$$

Assume that the ballot B_i is such that $H_i^{(1)} = \{x_2\}$, $H_i^{(-1)} = \{x_1\}$ and $H_i^{(0)} = X \backslash \{x_1, x_2\}$. Then $W_2 = W(B_i, B_{-i}) = \{x_2\}$.

These two sets of winners W_1 and W_2 are completely different as $W_1 \cap W_2 = \varnothing$, and the statement is proven.

So even only one voter (who knows all other ballots) can cardinally change the set of winners. Of course, it is much easier to do this when he knows the ballots of other voters.

5 Computational Complexity

The overall complexity is $O(nm + n \log n)$, where $m = \#I$ is the number of voters and $n = \#X$ is the number of proposals.

The computational complexity of the FTV method is quite efficient and can be broken down into three main stages.

1. **Score Aggregation:** $O(nm)$. For each of the n proposals, the system must sum the scores from all m voters. This is the most intensive part of the process if the number of voters is very large. The complexity is linear with respect to both proposals and voters.
2. **Project Ranking:** $O(n \log n)$. After the scores are tallied, the n proposals must be sorted from highest to lowest score. Using an efficient sorting algorithm, this step has a log-linear complexity relative to the number of proposals.
3. **Winner Selection:** $O(n)$. The final step involves a single pass through the sorted list of n proposals to select winners based on the available budget. This is a simple linear operation.

It should be noted that we do not consider here any particular privacy-preserving voting protocol that is needed to provide the secrecy of ballots (that is beyond the scope of this paper). After introducing such a protocol, the overall complexity will also include the complexity of the chosen privacy-preserving voting protocol [29,30].

6 Practical Deployment

The proposed FTV model is implemented in the Cardano treasury system—the Catalyst Project [1]. The Catalyst voting system is designed to foster decentralized innovation within the Cardano ecosystem. It empowers the community to propose, evaluate, and vote on projects that aim to enhance the blockchain ecosystem and address real-world challenges.

In each voting round, called a Fund, community members submit ideas for funding ranging from technical improvements to creative and business initiatives. The proposals are reviewed and rated by the community to ensure quality and feasibility. ADA holders participate in the voting process using the Catalyst Voting app. Voting power is proportional to the amount of ADA held, ensuring that stakeholders have a say in the network's development. Based on the voting results, the selected proposals receive funding from the treasury to bring their ideas to life.

The system ensures that decision-making is distributed among the community, aligning with Cardano's principles. All proposals, reviews, and voting outcomes are publicly accessible, fostering trust and accountability. Participants, including voters and reviewers, are rewarded in ADA for their contributions, encouraging active involvement.

Project Catalyst has already funded numerous initiatives, making it one of the largest decentralized innovation platforms in the blockchain space. Table 4 presents statistics on the recent Catalyst funds.

During all time, 2,161 proposals were funded by using the Cardano treasury system. 1,196

Table 4 The statistics on Catalyst project [31]

Fund #	Total Fund, ADA	Proposals' Fund, ADA	Proposals submitted	Proposals funded	Total votes cast	Active voting ADA
Fund8	16,000,000	12,800,000	1 134	367	237 485	1.2B
Fund9	16,000,000	12,800,000	1 166	207	364 288	1.8B
Fund10	50,000,000	46,590,000	1 467	262	409 749	2.3B
Fund11	50,000,000	46,500,000	920	300	307 698	2.2B
Fund12	50,000,000	46,500,000	1 205	258	291 737	2.3B
Fund13	50,000,000	46,500,000	1 639	199	310 597	2.5B

of them are already completed.

The limitations of the proposed FTV model in practical scenarios include the following:

- The voting system must be deployed on a secure blockchain that guarantees persistence and liveness;
- A privacy-preserving protocol must be used to secure votes and provide the necessary properties of the system;
- To be able to participate in voting, a voter must have a minimum required amount of stake (for example, in Catalyst, it was 500 ADA when the system was launched, and now it is equal to 25 ADA).

The selection of concrete values for these parameters (like the vote range, the total budget, or individual project budgets) is a system-specific decision. It would depend on the particular cryptocurrency governance ecosystem, its economic realities, community preferences, the nature of the projects being funded, and the desired level of expressiveness in voting. These values would typically be determined during the design and deployment phase of a specific treasury or governance system.

7 Conclusion

We gave a formal description of the proposed voting system, as well as an initial analysis of its properties. We called this system the fuzzy threshold voting. The main results are the following:

1) We presented a mathematical model for the FTV voting system. The main difference between this system from the yes-no-abstain voting system is that the voting procedure depends not only on the score of candidates but also on some other factor. In our case, this factor is the amount of money requested by each proposal.

2) We demonstrated that the presented FTV system satisfies one of the most important properties of the voting systems—the monotone rule.

3) We proved several statements regarding possible voting strategies.

4) We analyzed the importance of privacy-preserved voting and demonstrated that even a single voter can completely change the set of winners if he knows the voting ballots of other voters.

Data availability Data will be made available on request.

Compliance with Ethical Standards For this type of study, informed consent is not required.

Conflict of Interest The authors declare no conflict of interest.

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